# Two hybrid models for dependent death times of couple: a common shock approach

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**Abstract**

We combine two recent credit risk models with the Marshall-Olkin setup to capture the dependence structure of bivariate survival functions. The main advantage of this approach is to handle fatal shock events in the dependence structure since these two credit risk models allow to match the time of death of an individual with a catastrophe time event. We also provide a methodology for adding other sources of dependency in our approach. In such setup, we derive the no-arbitrage prices of some common life insurance product for coupled lives. We demonstrate the performance of our method by investigating Sibuya's dependence function. Calibration is done on the data of joint life contracts from a Canadian company.

***Keywords***— Credit risk, Dependence structure, Fatal shock events, Common life insurance

## 1 Introduction

### 1.1 Context

A joint membership contract in life insurance is a policy taken out by members of a married couple. This contract allows holders to protect themselves against their risk of death. However, the co-insureds can choose at the time of subscription the event that may result in the termination of the contract. The latter revolves around two options. Either on the death of one of the members of the couple, the capital thus accrued returns to the beneficiary of the contract, who in most cases is the surviving spouse. Either on the death of the next spouse and in this case, the contract remains

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open after the death of one of the co-members. However, even if this technique, allowing to open a single contract in the name of two people, presents a high number of advantages on the side of the insured, it is not so on the side of the insurer in a specific context linked the estimate of the couple's probability of death. Most models for estimating these probabilities are based on the independence hypothesis of the joint death times. Nevertheless, it has been shown that there is a dependency structure in the joint law of these times (see e.g., [5] and the literature their in). This dependence can be explained by the so-called broken-heart syndrome or by some others phenomena such as catastrophic exogenous events which may lead to the simultaneous death of the spouses.

Following the suggestion of Artzner and Delbaen [1], Milevsky and Promislow [23] is the first paper to recognize a parallel between reduced-form credit risk models and continuous-time mortality. The subsequent literature builds on this parallel to produce a variety of continuous-time mortality models that fits well the observed mortality patterns and where actuarial quantities and mortality derivatives can be more or less easily priced. Some existing work using the credit risk approach in life insurance modelling can be seen for example in [4], [21], etc...

This parallelism also allows many authors to model joint mortality of coupled lives. Dependence random time models have been developed in the literature of credit risk and life insurance. Some of these models are based on copulas (see e.g., [5, 27] ) and common shock approaches (see, e.g, [18, 17, 16] ) which allow handling correlation between lifetimes through the hypothesis that dependency is caused by external shock affecting both spouses. Others are based on the markovian approaches (see e.g., [12, 25] ) for capturing the state change in the couple's lifetimes. This last allows taking into account the broken-heart syndrome. Among the most successful attempts, [14] develop a bivariate model for mortality by combining markovian copula approaches to handle the set of effects related to the broken-heart syndrome. Their idea is based on the disadvantages of both the markovian and copula approaches. Indeed, using copula is difficult when implementing dynamic modelling while the markovian approach fails to show the dependency between the lifetimes of a couple([14]). Two recent contributions ([9, 10]) extend the literature of joint mortality modelling by introducing a modelling framework in which the dependence structure is due to a copula function and a random fatal shock that causes the death of both annuitants. This second source of dependence is known as Marshall-Olkin. However, the modelling framework of [9, 10] is valid only when the mortality is thought as a time-independent phenomenon.

In this paper, we insist on the parallel between reduced-form credit risk models and stochastic mortality. We build two hybrid general class of models by combining the **Generalized Cox** [11] and **Generalized Jiao** and Li models [11] with Marshall-Olkin approach. Our modelling framework turns out to be quite flexible and easy to apply, as new models incorporating Marshall-Olkin-like dependence structure can be built on top of already existing models. In addition, it accommodates the stochastic nature of mortality in a straightforwardly manner. No-arbitrage pricing of classical life-insurance products on two heads is also relatively easy.

The layout of the paper is as follows. In the following subsection, we describe the theoretical background that leads to our modelling framework. In Section 2, we construct a general model of joint mortality based on the approach of Marshall-Olkin. We then specialize the two hybrid models and derive the survival functions. The no-arbitrage pricing of common life-insurance products under the Generalized Cox model is carried out in Section 3. Section 4 reports a numerical application based on the well-known Canadian dataset and Section 5 concludes.

### 1.2 Motivations

All the literature of the parallelism of these two approaches shares the same spirit, as explained below.

The building block of all the models is a positive random time $\tau^x$, defined in a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$, which represents the (random) remaining lifetime of an individual aged $x$. This random time is often thought to be the first passage time an increasing (absolutely) continuous stochastic process $\Gamma^x$ ($\mathbb{F}$-adapted) to a random barrier $\Theta^x$ (usually supposed to be an exponential with parameter 1 and independent to $\mathbb{F}$), i.e.,

$$\tau^x := \inf\{t \geq 0 : \Gamma^x_t > \Theta^x\}. \tag{1.1}$$

Note that this random time $\tau^x$, known as standard Cox one (see [15]), avoids all $\mathbb{F}$-stopping times (i.e., $\mathbb{P}(\tau^x = \nu) = 0$ a.s., for all $\mathbb{F}$-stopping time $\nu$).

As mentioned in Protter et al [26], difficulties arise when one deals with two or more random lifetimes (e.g, here with two random lifetimes $\tau^x$ and $\tau^y$) with dependency induced by an external shock event. Hence, they add some modifications along the lines of an extension of the Marshall and Olkin [22] approach.

Their model appeals to the theory of progressive enlargement of filtration although the authors did not insist on this framework. It consists in considering three $\mathbb{F}$-conditional independent Cox random times $\mu^1$, $\mu^2$ and $\mu^3$ and enlarge $\mathbb{F}$ with them to obtain a filtration $\mathbb{G}$. Then define the random lifetimes $\tau^x$ and $\tau^y$ as :

$$\tau^x := \mu^1 \wedge \mu^3 \quad \text{and} \quad \tau^y := \mu^2 \wedge \mu^3.$$

Since the time $\mu^3$ of the external shock event is totally inaccessible and avoids all the $\mathbb{F}$-stopping, both the random times $\tau^x$ and $\tau^y$ are also totally inaccessible and avoid all $\mathbb{F}$-stopping times. In addition, this model fails to take proper care over the non-fatal shocks which may also contribute to the death of both spouses. By means it should be interesting to take into account some external arrival shocks that may not be so fatal but increase the exposure of death of the spouse.

These last observations shape our thinking to build a model with several external shock events with times of occurrence being $\mathbb{F}$-stopping times. Hence, we construct some dependence random times that do not avoid $\mathbb{F}$-stopping times and which can be accessible. Our construction is a combination of the so-called generalized Cox model (see [11]) and the Marshall and Olkin approach.

The main difference with Protter et al [26] approach is the way to construct the random fatal shock time which leads to some different results. Moreover, the novelty of our construction lies in the fact that it allows obtaining random lifetimes which can be accessible.

## 2 Construction framework

In what follows, we omit the dependence on ages ($x$ and $y$) to keep notation as simple as possible.

We consider the filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a complete and continuous on right filtration, and $\mathcal{G}$ a $\sigma$-algebra satisfying $\mathcal{F}_\infty \subset \mathcal{G}$.
Let the following random times $\tau^f := \theta \wedge \xi^f$ and $\tau^m := \theta \wedge \xi^m$ where $\theta$ is the fatal shock time constructed as either in **Generalized Cox** or **Generalized Jiao and Li** framework and the random times $\xi^f$ and $\xi^m$ are two standard Cox random times, i.e., $\xi^f := \inf\{t \geq 0 : F_t > \Theta^f\}$ and $\xi^m := \inf\{t \geq 0 : M_t > \Theta^m\}$ where $F$ and $M$ are two increasing $\mathbb{F}$-adapted continuous processes with $F_0 = M_0 = 0$, $F_t < \infty$, $M_t < \infty$ for all $t$, and $F_\infty = \infty$, $M_\infty = \infty$ and the independent random variables $\Theta^f$ and $\Theta^m$ are unit exponential random ones which are independent of $\mathbb{F}$.
We introduce the two right-continuous increasing processes $A_t^f = \mathbb{1}_{\{\tau^f \leq t\}}$ and $A_t^m = \mathbb{1}_{\{\tau^m \leq t\}}$ associated respectively with $\tau^f$ and $\tau^m$ and we denote by $\mathbb{A}^f = (\mathcal{A}_t^f)_{t\geq 0}$ and $\mathbb{A}^m = (\mathcal{A}_t^m)_{t\geq 0}$ the (completed and right-continuous) filtrations generated by these processes. We denote by $\mathbb{G}$ the global information so that $\mathbb{G} = \mathbb{A}^f \vee \mathbb{A}^m \vee \mathbb{F}$.

### 2.1 The generalized Cox model framework

Here, we recall the construction of the generalized Cox model as it is done in [11] which constitutes a way to construct our fatal shock time.

Let some arrival external shock events with times of occurrence $(\tau_i)_i$ being a strictly increasing sequence of $\mathbb{F}$-stopping times and corresponding to the jump times of an increasing càdlàg $\mathbb{F}$-adapted process $K$ such that $K_0 = 0$, $K_t < \infty$, for all $t \geq 0$ and $K_\infty = \infty$. We define the fatal shock time $\theta$ as the first time that the process $K$ hits a level which is a positive random variable $\Theta$ independent of the filtration $\mathbb{F}$, i.e.,

$$\theta := \inf\{t \geq 0 : K_t > \Theta\}. \tag{2.1}$$

It is shown in [11] that $\{\theta = \tau_i\} = \{K_{\tau_i -} < \Theta \leq \tau_i\}$ and then

$$\mathbb{P}(\tau = \tau_i|\mathcal{F}_t) \quad = \quad \mathbb{E}\left[e^{-K_{\tau_i -}}(1 - e^{-\Delta K_{\tau_i}})|\mathcal{F}_t\right], \forall t \geq 0\,.$$

Hence, the random time $\theta$ does not avoid all $\mathbb{F}$-stopping.
Note that several examples of $K$ have been studied in [11] and one can use some of them in our model.

#### 2.1.1 The $\mathbb{F}$-conditional survival law and the joint probability of $\tau^f$ and $\tau^m$

**Lemma 2.1** *For all $t \geq 0$, one has*

$$\mathbb{P}(\{\tau^f = \tau^m = \tau_i\}|\mathcal{F}_t) = \mathbb{E}\left[e^{-F_{\tau_i}}e^{-M_{\tau_i}}[e^{-K_{\tau_i -}}(1 - e^{-\Delta K_{\tau_i}})]|\mathcal{F}_t\right], \forall i \geq 1. \tag{2.2}$$

Proof: From the definition of $\tau^f$ and $\tau^m$, one has the following equality

$$\{\tau^f = \tau^m = \tau_i\} = \{\xi^f > \theta, \xi^m > \theta, \theta = \tau_i\}.$$

Hence, by using the definition of $\xi^f$ and $\xi^m$ then

$$\mathbb{P}(\xi^f > \theta, \xi^m > \theta, \theta = \tau_i|\mathcal{F}_\infty) = \mathbb{P}(F_{\tau_i} < \Theta^f, M_{\tau_i} < \Theta^m, K_{\tau_i-} < \Theta \le \tau_i|\mathcal{F}_\infty).$$

Since $\Theta^f$, $\Theta^m$ and $\Theta$ are mutually independent and are independent of $\mathcal{F}_\infty$, one obtains

$$\mathbb{P}(\tau^f = \tau^m = \tau_i|\mathcal{F}_\infty) = e^{-F_{\tau_i}} e^{-M_{\tau_i}} [e^{-K_{\tau_i-}}(1 - e^{-\Delta K_{\tau_i}})].$$

□

**Lemma 2.2** *For any* $t_1, t_2, t \in \mathbb{R}^+$*, one has*

$$\mathbb{P}(\tau^f > t_1, \tau^m > t_2|\mathcal{F}_t) = \mathbb{E}\left[\exp\left(-F_{t_1} - M_{t_2} - K_{\max(t_1,t_2)}\right)\Big|\mathcal{F}_t\right]. \tag{2.3}$$

*In particular, if* $\max(t_1, t_2) \le t$*, one obtains*

$$\mathbb{P}(\tau^f > t_1, \tau^m > t_2|\mathcal{F}_t) = \exp\left(-F_{t_1} - M_{t_2} - K_{\max(t_1,t_2)}\right). \tag{2.4}$$

Proof: For all $t_1, t_2, t \in \mathbb{R}^+$,

$$\begin{aligned}\mathbb{P}(\tau^f > t_1, \tau^m > t_2|\mathcal{F}_\infty) &= \mathbb{P}(\xi^f > t_1, \xi^f > t_2, \theta > \max(t_1, t_2)|\mathcal{F}_\infty)\\ &= \mathbb{P}(F_{t_1} < \Theta^f, M_{t_2} < \Theta^m, K_{\max(t_1,t_2} < \Theta|\mathcal{F}_\infty)\\ &= \exp\left(-F_{t_1} - M_{t_2} - K_{\max(t_1,t_2)}\right).\end{aligned}$$

□

### 2.1.2 Some closed forms

We consider the case where the filtration is generated by $K$ and two independent Brownian motions $W^1$ and $W^2$ which are independent of $K$ with $F_t = \int_0^t \gamma_s^f ds$ and $M_t = \int_0^t \gamma_s^m ds$ where $\gamma^f$ is a non-negative $\mathbb{F}^{W^1}$-adapted process and $\gamma^m$ is non-negative $\mathbb{F}^{W^2}$-adapted.

We have for any $t \le \min(t_1, t_2)$

$$\mathbb{E}\left[e^{-F_{t_1}} e^{-M_{t_2}} e^{-K_{\max(t_1,t_2)}}\Big|\mathcal{F}_t\right] = \mathbb{E}\left[\mathbb{E}\left[e^{-F_{t_1}} e^{-M_{t_2}} e^{-K_{\max(t_1,t_2)}}\Big|\mathcal{F}_t^{W^1} \vee \mathcal{F}_t^{W^2} \vee \mathcal{F}_T^K\right]\Big|\mathcal{F}_t\right], \text{for } \max(t_1, t_2) \le T.$$

Hence, since $K_{\max(t_1,t_2)} \in \mathcal{F}_T^K \subset \mathcal{F}_t^{W^1} \vee \mathcal{F}_t^{W^2} \vee \mathcal{F}_T^K$ then,

$$\mathbb{E}\left[e^{-F_{t_1}} e^{-M_{t_2}} e^{-K_{\max(t_1,t_2)}} \middle| \mathcal{F}_t\right] = \mathbb{E}\left[e^{-K_{\max(t_1,t_2)}} \mathbb{E}\left[e^{-F_{t_1}} e^{-M_{t_2}} \middle| \mathcal{F}_t^{W^1} \vee \mathcal{F}_t^{W^2} \vee \mathcal{F}_T^K\right] \middle| \mathcal{F}_t\right].$$

Due to the fact that $\mathcal{F}_t^{W^1} \vee \mathcal{F}_t^{W^2}$ is independent of $\mathcal{F}_T^K$, then $F_{t_1} \in \mathcal{F}_t^{W^1} \vee \mathcal{F}_t^{W^2}$ and $M_{t_2} \in \mathcal{F}_t^{W^1} \vee \mathcal{F}_t^{W^2}$ are independent of $\mathcal{F}_T^K$ and then

$$\begin{aligned}
\mathbb{E}\left[e^{-F_{t_1}} e^{-M_{t_2}} e^{-K_{\max(t_1,t_2)}} \middle| \mathcal{F}_t\right] &= \mathbb{E}\left[e^{-K_{\max(t_1,t_2)}} \mathbb{E}\left[e^{-F_{t_1}} e^{-M_{t_2}} \middle| \mathcal{F}_t^{W^1} \vee \mathcal{F}_t^{W^2}\right] \middle| \mathcal{F}_t\right] \\
&= \mathbb{E}\left[e^{-F_{t_1}} e^{-M_{t_2}} \middle| \mathcal{F}_t^{W^1} \vee \mathcal{F}_t^{W^2}\right] \mathbb{E}\left[e^{-K_{\max(t_1,t_2)}} \middle| \mathcal{F}_t\right]
\end{aligned}$$

where we have used the fact that $\mathbb{E}\left[e^{-F_{t_1}} e^{-M_{t_2}} \middle| \mathcal{F}_t^{W^1} \vee \mathcal{F}_t^{W^2}\right]$ is $\mathcal{F}_t$-measurable in the last equality. Using the same reasoning, we obtain

$$\begin{aligned}
\mathbb{E}\left[e^{-F_{t_1}} e^{-M_{t_2}} e^{-K_{\max(t_1,t_2)}} \middle| \mathcal{F}_t\right] &= \mathbb{E}\left[\mathbb{E}\left[e^{-F_{t_1}} e^{-M_{t_2}} \middle| \mathcal{F}_t^{W^1} \vee \mathcal{F}_t^{W^2} \vee \mathcal{F}_T^{W^2}\right] \middle| \mathcal{F}_t^{W^1} \vee \mathcal{F}_t^{W^2}\right] \mathbb{E}\left[e^{-K_{\max(t_1,t_2)}} \middle| \mathcal{F}_t^K\right] \\
&= \mathbb{E}\left[e^{-M_{t_2}} \mathbb{E}\left[e^{-F_{t_1}} \middle| \mathcal{F}_t^{W^1} \vee \mathcal{F}_t^{W^2} \vee \mathcal{F}_T^{W^2}\right] \middle| \mathcal{F}_t^{W^1} \vee \mathcal{F}_t^{W^2}\right] \mathbb{E}\left[e^{-K_{\max(t_1,t_2)}} \middle| \mathcal{F}_t^K\right] \\
&= \mathbb{E}\left[e^{-M_{t_2}} \mathbb{E}\left[e^{-F_{t_1}} \middle| \mathcal{F}_t^{W^1} \vee \mathcal{F}_t^{W^2}\right] \middle| \mathcal{F}_t^{W^1} \vee \mathcal{F}_t^{W^2}\right] \mathbb{E}\left[e^{-K_{\max(t_1,t_2)}} \middle| \mathcal{F}_t^K\right] \\
&= \mathbb{E}\left[e^{-M_{t_2}} \mathbb{E}\left[e^{-F_{t_1}} \middle| \mathcal{F}_t^{W^1}\right] \middle| \mathcal{F}_t^{W^1} \vee \mathcal{F}_t^{W^2}\right] \mathbb{E}\left[e^{-K_{\max(t_1,t_2)}} \middle| \mathcal{F}_t^K\right] \\
&= \mathbb{E}\left[e^{-F_{t_1}} \middle| \mathcal{F}_t^{W^1}\right] \mathbb{E}\left[e^{-M_{t_2}} \middle| \mathcal{F}_t^{W^1} \vee \mathcal{F}_t^{W^2}\right] \mathbb{E}\left[e^{-K_{\max(t_1,t_2)}} \middle| \mathcal{F}_t^K\right] \\
&= \mathbb{E}\left[e^{-F_{t_1}} \middle| \mathcal{F}_t^{W^1}\right] \mathbb{E}\left[e^{-M_{t_2}} \middle| \mathcal{F}_t^{W^2}\right] \mathbb{E}\left[e^{-K_{\max(t_1,t_2)}} \middle| \mathcal{F}_t^K\right].
\end{aligned}$$

Note that some closed forms can be obtained for the first two quantities in the last equality when one deals with $F$ and $M$ related to some specific processes such as affine or polynomial processes. Hence, we only consider these notations, for now, $h_t^1(0,t_1) := \mathbb{E}\left[e^{-F_{t_1}} \middle| \mathcal{F}_t^{W^1}\right]$ and $h_t^2(0,t_2) := \mathbb{E}\left[e^{-M_{t_2}} \middle| \mathcal{F}_t^{W^2}\right]$. Hence, the $\mathbb{F}$-conditional survival joint probability is given by

$$\mathbb{P}(\{\tau^f > t_1, \tau^m > t_2\} | \mathcal{F}_t) = h_t^1(0,t_1) h_t^2(0,t_2) \mathbb{E}\left[e^{-K_{\max(t_1,t_2)}} \middle| \mathcal{F}_t^K\right]. \tag{2.5}$$

For example, if $\gamma^f$ and $\gamma^m$ are affine processes, then one has for $\min(t_1,t_2) \geq t$

$$h_t^1(0,t_1) = e^{-\int_0^t \gamma_s^f ds} e^{A_t^1(t_1) - B_t^1(t_2)\gamma_t^f},$$

and

$$h_t^2(0,t_2) = e^{-\int_0^t \gamma_s^m ds} e^{A_t^2(t_1) - B_t^2(t_2)\gamma_t^m},$$

where for $j \in \{1,2\}$, $A^j$ and $B^j$ are differentiable functions with $A_{t_j}^j(t_j) = 0$ and $B_{t_j}^j(t_j) = 0$ and verify generalized Riccati ODEs (see [7]).

**Comment 2.3** Some closed forms can be obtaining when playing with the type of the process $K$.

For instance one can choose $K$ to be a subordinator, a marked point process, a shot noise...

**When the process $K$ is a subordinator:** we consider now the process $K$ to be a null drift Lévy subordinator with Lévy's measure $\nu$. Hence, $\eta_u := e^{-K_u}e^{\phi(1)u}$, $\forall u \geq 0$ (where we $\phi(1) := \int_{\mathbb{R}^+}(1-e^{-x})\,\nu(dx)$) is an $\mathbb{F}^K$-martingale. Therefore $\mathbb{E}[e^{-K_u}\mathcal{F}_t^K] = \eta_t e^{-\phi(1)u}$, for all $u \geq t$ and by consequence, for all $t_1, t_2 \in \mathbb{R}^+$ such that $\min(t_1,t_2) \geq t \geq 0$, one has

$$\mathbb{E}[e^{-K_{\max(t_1,t_2)}}|\mathcal{F}_t^K] = \eta_t e^{-\phi(1)\max(t_1,t_2)}.$$

Therefore, for $\min(t_1,t_2) \geq t$, the joint conditional survival probability of $\tau^f$ and $\tau^m$ has the following form:

$$\mathbb{P}(\tau^f > t_1, \tau^m > t_2|\mathcal{F}_t) = h_t^1(0,t_1)h_t^2(0,t_2)\eta_t e^{-\phi(1)\max(t_1,t_2)}. \tag{2.6}$$

## 2.2 The Generalized Jiao and Li framework

The so-called generalized Jiao and Li model (see [11]) is an extension of the model in [13] developed in the sovereign risk modelling to capture the impacts of some possible arrival shock events on the prices of the defaultable claims which may induce some jumps in these prices. In the generalized Jiao and Li model, the fatal shock time can be constructed as follows.

We consider the (supposed) increasing càdlàg process $X$ such that $X_0 = 0$, $X_\infty = \infty$, independent of $\mathbb{F}$ and an increasing sequence of $\mathbb{F}$-adapted stopping times $(\tau_i)_{i\geq 1}$. One denotes by $\Psi$ the increasing deterministic function with $\Psi(0) = 0$ and $\Psi(\infty) = \infty$ such that $\mathbb{P}(X_u \leq 1) = e^{-\Psi(u)}$. One denotes by $\mathbb{F}^X$ the natural filtration of $X$. We are given the following form of the fatal shock time $\theta$

$$\theta = \tau_i \quad \text{on} \quad \{X_{\tau_{i-1}} \leq 1 < X_{\tau_i}\}, \text{ for } i \geq 1 \tag{2.7}$$

and

$$\xi := \inf\{t \geq 0 : \Gamma_t > \Theta\} \tag{2.8}$$

where $\Gamma$ is an increasing $\mathbb{F}$-adapted continuous process with $\Gamma_0 = 0$, $\Gamma_t < \infty$ for all $t$, and $\Gamma_\infty = \infty$ and $\Theta$ a random variable, independent of $\mathbb{F}$ and $\mathbb{F}^X$, with a unit exponential law.

### 2.2.1 The $\mathbb{F}$-conditional survival law and the joint probability of $\tau^f$ and $\tau^m$

The following trivial equality could frequently be used in this work

$$\sum_{i\geq 1} \mathbb{1}_{\{\tau_i > t \geq \tau_{i-1}\}} e^{-U(\tau_{i-1})} = \exp\left(-\sum_{i\geq 1}\mathbb{1}_{\{\tau_i > t \geq \tau_{i-1}\}} U(\tau_{i-1})\right), \tag{2.9}$$

for any function $U : \mathbb{R}^+ \to \mathbb{R}$.

**Lemma 2.4** *For all $t \geq 0$, one has*

$$\mathbb{P}(\tau^f = \tau^m = \tau_i|\mathcal{F}_t) = \mathbb{E}[e^{-(F_{\tau_i}+M_{\tau_i})}\left\{e^{-\Psi(\tau_{i-1})} - e^{-\Psi(\tau_i)}\right\}|\mathcal{F}_t],\ \forall i \geq 1. \tag{2.10}$$

Proof: From the definition of $\tau^f$ and $\tau^m$, one has the following equality

$$\{\tau^f = \tau^m = \tau_i\} = \{\xi^f > \tau_i, \xi^m > \tau_i, X_{\tau_{i-1}} \leq 1 < X_{\tau_i}\}.$$

Hence, by using the definition of $\xi^f$ and $\xi^m$ as well as the fact that $(\tau_i)_i$ are $\mathbb{F}$-stopping times, hence are $\mathcal{F}_\infty$-measurable random variables and that the random variable $X_{\tau_i}$ is, for any $i$, $\mathcal{F}_\infty \vee \mathcal{F}^X_\infty$ measurable, one obtains

$$\begin{aligned}\mathbb{P}(\xi^f > \tau_i, \xi^m > \tau_i, X_{\tau_{i-1}} \leq 1 < X_{\tau_i}|\mathcal{F}_\infty) =&\mathbb{E}\left[\mathbb{P}(\xi^f > \tau_i|\mathcal{F}_\infty \vee \mathcal{F}^X_\infty)\mathbb{P}(\xi^m > \tau_i|\mathcal{F}_\infty \vee \mathcal{F}^X_\infty)\mathbb{1}_{\{X_{\tau_{i-1}} \leq 1 < X_{\tau_i}\}}|\mathcal{F}_\infty\right]\\ =&\mathbb{E}\left[\mathbb{P}(F_{\tau_i} < \Theta^f|\mathcal{F}_\infty \vee \mathcal{F}^X_\infty)\mathbb{P}(M_{\tau_i} < \Theta^m|\mathcal{F}_\infty \vee \mathcal{F}^X_\infty)\mathbb{1}_{\{X_{\tau_{i-1}} \leq 1 < X_{\tau_i}\}}|\mathcal{F}_\infty\right]\end{aligned}$$

where the first equality requires the tower property and the independence of $\xi^f$ and $\xi^m$.
Since $\Theta^f$, $\Theta^m$ and $X$ are mutually independent and are independent of $\mathcal{F}_\infty$, then using the fact that $F_{\tau_i} \in \mathcal{F}_\infty$ and $M_{\tau_i} \in \mathcal{F}_\infty$ leads to

$$\mathbb{P}(F_{\tau_i} < \Theta^f|\mathcal{F}_\infty \vee \mathcal{F}^X_\infty) = \mathbb{P}(F_{\tau_i} < \Theta|\mathcal{F}_\infty) = e^{-F_{\tau_i}} \tag{2.11}$$

and

$$\mathbb{P}(M_{\tau_i} < \Theta^m|\mathcal{F}_\infty \vee \mathcal{F}^X_\infty) = \mathbb{P}(M_{\tau_i} < \Theta|\mathcal{F}_\infty) = e^{-M_{\tau_i}}. \tag{2.12}$$

Therefore, it follows

$$\begin{aligned}\mathbb{P}(\xi^f > \tau_i, \xi^m > \tau_i, X_{\tau_{i-1}} \leq 1 < X_{\tau_i}|\mathcal{F}_\infty) =&e^{-F_{\tau_i}}e^{-M_{\tau_i}}\mathbb{P}(X_{\tau_{i-1}} \leq 1 < X_{\tau_i}|\mathcal{F}_\infty)\\ =&e^{-F_{\tau_i}}e^{-M_{\tau_i}}\{\mathbb{P}(X_{\tau_{i-1}} \leq 1|\mathcal{F}_\infty) - \mathbb{P}(X_{\tau_i} \leq 1|\mathcal{F}_\infty)\}\\ =&e^{-F_{\tau_i}}e^{-M_{\tau_i}}\left\{e^{-\Psi(\tau_{i-1})} - e^{-\Psi(\tau_i)}\right\}.\end{aligned} \tag{2.13}$$

The last equality is due to the fact that the random variables $\tau_i$ are $\mathcal{F}_\infty$-measurable and the process $X$ is independent of $\mathcal{F}_\infty$.
Therefore

$$\mathbb{P}(\tau^f = \tau^m = \tau_i|\mathcal{F}_t) = \mathbb{E}[e^{-(F_{\tau_i}+M_{\tau_i})}\left\{e^{-\Psi(\tau_{i-1})} - e^{-\Psi(\tau_i)}\right\}|\mathcal{F}_t].$$

□

**Lemma 2.5** *For any $t_1, t_2, t \in \mathbb{R}^+$, one has*

$$\mathbb{P}(\tau^f > t_1, \tau^m > t_2|\mathcal{F}_t) = \mathbb{E}\left[\exp\left(-\sum_{i=1}^{\infty}\mathbb{1}_{\{\tau_i \leq \max(t_1,t_2)\}}[\Psi(\tau_i) - \Psi(\tau_{i-1})] - (F_{t_1} + M_{t_2})\right)|\mathcal{F}_t\right]. \tag{2.14}$$

*In particular, if* $\max(t_1, t_2) \leq t$, *one obtains*

$$\mathbb{P}(\tau^f > t_1, \tau^m > t_2 | \mathcal{F}_t) = \exp\left( - \sum_{i=1}^{\infty} \mathbb{1}_{\{\tau_i \leq \max(t_1,t_2)\}} [\Psi(\tau_i) - \Psi(\tau_{i-1})] - (F_{t_1} + M_{t_2}) \right). \tag{2.15}$$

Proof: For all $t_1, t_2, t \in \mathbb{R}^+$,

$$\begin{aligned}\mathbb{P}(\tau^f > t_1, \tau^m > t_2 | \mathcal{F}_t) &= \sum_{i=1}^{\infty} \mathbb{P}(\theta > t_1, \xi^f > t_1, \theta > t_2, \xi^f > t_2, \theta = \tau_i | \mathcal{F}_t) \\ &= \sum_{i=1}^{\infty} \mathbb{P}(\tau_i > t_1, \tau_i > t_2, \xi^f > t_1, \xi^m > t_2, X_{\tau_{i-1}} \leq 1 < X_{\tau_i} | \mathcal{F}_t).\end{aligned}$$

By using the fact that $\tau_0 = 0$ (which implies that the set $\{\tau_0 > u\}$ is empty), one has

$$\begin{aligned}\mathbb{P}(\tau^f > t_1, \tau^m > t_2 | \mathcal{F}_t) &= \sum_{i=1}^{\infty} \mathbb{P}(\tau_i > \max(t_1, t_2) \geq \tau_{i-1}, \xi^f > t_1, \xi^m > t_2, X_{\tau_{i-1}} \leq 1 | \mathcal{F}_t) \\ &= \sum_{i=1}^{\infty} \mathbb{E}[\mathbb{1}_{\{\tau_i > \max(t_1,t_2) \geq \tau_{i-1}\}} \mathbb{P}(\xi^f > t_1 | \mathcal{F}_\infty \vee \mathcal{F}^X_\infty) \mathbb{P}(\xi^m > t_2 | \mathcal{F}_\infty \vee \mathcal{F}^X_\infty) \mathbb{1}_{\{X_{\tau_{i-1}} \leq 1\}} | \mathcal{F}_t] \\ &= \sum_{i=1}^{\infty} \mathbb{E}[\mathbb{1}_{\{\tau_i > \max(t_1,t_2) \geq \tau_{i-1}\}} e^{-F_{t_1}} e^{-M_{t_2}} \mathbb{1}_{\{X_{\tau_{i-1}} \leq 1\}} | \mathcal{F}_t],\end{aligned}$$

where we have used, in the last equality, the fact that $\Theta^f$ independent of $\Theta^m$ and together independent of $\mathbb{F} \vee \mathbb{F}^X$. This implies

$$\begin{aligned}\mathbb{P}(\tau^f > t_1, \tau^m > t_2 | \mathcal{F}_t) &= \sum_{i=1}^{\infty} \mathbb{E}[\mathbb{1}_{\{\tau_i > \max(t_1,t_2) \geq \tau_{i-1}\}} e^{-F_{t_1}} e^{-M_{t_2}} \mathbb{P}(X_{\tau_{i-1}} \leq 1 | \mathcal{F}_\infty) | \mathcal{F}_t] \\ &= \mathbb{E}\left[ \sum_{i=1}^{\infty} \mathbb{1}_{\{\tau_i > \max(t_1,t_2) \geq \tau_{i-1}\}} e^{-(F_{t_1} + M_{t_2})} e^{-\Psi(\tau_{i-1})} | \mathcal{F}_t \right] \\ &= \mathbb{E}\left[ \exp\left( - \sum_{i=1}^{\infty} \mathbb{1}_{\{\tau_i > \max(t_1,t_2) \geq \tau_{i-1}\}} \Psi(\tau_{i-1}) \right) e^{-(F_{t_1} + M_{t_1})} | \mathcal{F}_t \right] \\ &= \mathbb{E}\left[ \exp\left( - \sum_{i=0}^{\infty} \mathbb{1}_{\{\tau_i \leq \max(t_1,t_2)\}} \Psi(\tau_i) + \sum_{i=1}^{\infty} \mathbb{1}_{\{\tau_i \leq \max(t_1,t_2)\}} \Psi(\tau_{i-1}) \right) e^{-(F_{t_1} + M_{t_2})} | \mathcal{F}_t \right].\end{aligned}$$

Since $\Psi(\tau_0) = 0$, one has

$$\sum_{i=0}^{\infty} \mathbb{1}_{\{\tau_i \leq \max(t_1,t_2)\}} \Psi(\tau_i) = \sum_{i=1}^{\infty} \mathbb{1}_{\{\tau_i \leq \max(t_1,t_2)\}} \Psi(\tau_i).$$

Hence, it follows

$$\mathbb{P}(\tau^f > t_1, \tau^m > t_2|\mathcal{F}_t) = \mathbb{E}\left[\exp\left(-\sum_{i=1}^{\infty} \mathbb{1}_{\{\tau_i \leq \max(t_1,t_2)\}}[\Psi(\tau_i) - \Psi(\tau_{i-1})] - (F_{t_1} + M_{t_2})\right)|\mathcal{F}_t\right].$$

□

### 2.2.2 Some specific cases

**The case where $F$ and $M$ are deterministic** We Assume that $F_t = F(t) := \int_0^t \gamma^f(s)ds$ and $M_t = M(t) := \int_0^t \gamma^m(s)ds$ with $F(\infty) = \infty$, $M(\infty) = \infty$, and where $F$ and $M$ are deterministic positive functions. We assume to know, for any $i$, the $\mathbb{F}$-compensator of $\tau_i$, i.e., the $\mathbb{F}$-predictable increasing process $J^i$, with $J_0^i = 0$, such that

$$(\mathbb{1}_{\{\tau_i \leq t\}} - J^i_{t\wedge\tau_i}, t \geq 0)$$

is an $\mathbb{F}$-martingale.

**Lemma 2.6** *For any $t_1, t_2, t \in \mathbb{R}^+$, the joint survival probability of $\tau^f$ and $\tau^m$ is given by*

$$\mathbb{P}(\tau^f > t_1, \tau^m > t_2) = \exp\left(-[F(t_1) + M(t_2)]\right)\left(1 - \mathbb{E}\left[\sum_{i=0}^{\infty}\int_0^{\max(t_1,t_2)} \mathbb{1}_{\{s<\tau_i\}}\left(e^{-\Psi(\tau_i)} - e^{-\Psi(s)}\right)dJ_s^{i+1}\right]\right).$$

Proof: The proof is based in [11], we describe the different steps.
In this case, the joint survival probability is given by

$$\mathbb{P}(\tau^f > t_1, \tau^m > t_2) = \exp\left(-[F(t_1) + M(t_2)]\right)\mathbb{E}\left[\exp\left(-\sum_{i=1}^{\infty} \mathbb{1}_{\{\tau_i \leq \max(t_1,t_2)\}}[\Psi(\tau_i) - \Psi(\tau_{i-1})]\right)\right].$$

Take $u := max(t_1, t_2)$, then the process

$$Q_u := \exp\left(-\sum_{i=1}^{\infty} \mathbb{1}_{\{\tau_i \leq u\}}[\Psi(\tau_i) - \Psi(\tau_{i-1})]\right) \tag{2.16}$$

is decreasing, hence is a supermartingale. Using (2.9), we get

$$Q_u = \sum_{i=0}^{\infty} \mathbb{1}_{\{\tau_i \leq u < \tau_{i+1}\}} e^{-\Psi(\tau_i)} = \sum_{i=0}^{\infty}\left(\mathbb{1}_{\{\tau_i \leq u\}} e^{-\Psi(\tau_i)} - \mathbb{1}_{\{\tau_{i+1} \leq u\}} e^{-\Psi(\tau_i)}\right) = \sum_{i=0}^{\infty}\left(Q_u^{1,i} - Q_u^{2,i}\right)$$

where the processes $Q^{1,i}$ and $Q^{2,i}$ are the ones defined $Q_u^{1,i} := \mathbb{1}_{\{\tau_i \leq u\}} e^{-\Psi(\tau_i)}$ and $Q_u^{2,i} := \mathbb{1}_{\{\tau_{i+1} \leq u\}} e^{-\Psi(\tau_i)}$. Therefore, $Q$ admits the decomposition $Q = m^Q - \zeta$, where $m^Q$ is a martingale and $\zeta$ is the predictable process given as

$$\zeta_u = \sum_{i=0}^{\infty}\left(\int_0^u \mathbb{1}_{\{\tau_i<s\}} e^{-\Psi(\tau_i)} dJ_s^{i+1} - \int_0^u e^{-\Psi(s)} dJ_s^{i}\right).$$

Since $M^{i+1} = A^{i+1} - J^{i+1}$ is a martingale with $M_0^{i+1} = 0$ and $A_{\tau_i}^{i+1} = 0$, we obtain

$$\mathbb{E}[J_{\tau_i}^{i+1}] = \mathbb{E}[A_{\tau_i}^{i+1} - M_{\tau_i}^{i+1}] = -\mathbb{E}[M_{\tau_i}^{i+1}] = 0$$

which implies that $J_{\tau_i}^{i+1} = 0$ and then due to the increasing property of $J^{i+1}$, $J_t^{i+1} = 0$ on $\{t \leq \tau_i\}$. This shows that the support of $J^{i+1}$ is $[\tau_i, \tau_{i+1}]$.
Hence, it follows that $\int_0^u \mathbb{1}_{\{\tau_i<s\}} e^{-\Psi(\tau_i)} dJ_s^{i+1} = \int_0^u e^{-\Psi(\tau_i)} dJ_s^{i+1}$, and then

$$\zeta_u = \sum_{i=0}^{\infty}\int_0^u e^{-\Psi(\tau_i)} dJ_s^{i+1} - \sum_{i=1}^{\infty}\int_0^u e^{-\Psi(s)} dJ_s^{i} = \sum_{i=0}^{\infty}\int_0^u \left(e^{-\Psi(\tau_i)} - e^{-\Psi(s)}\right) dJ_s^{i+1}$$

where we have used the fact that $J^0 = 0$.
Due to the form of the support of $J^{i+1}$, we have $\zeta_u = \sum_{i=0}^{\infty}\int_0^u \mathbb{1}_{\{s<\tau_i\}}\left(e^{-\Psi(\tau_i)} - e^{-\Psi(s)}\right) dJ_s^{i+1}$, hence, since $e^{-\Psi(\tau_i)} - e^{-\Psi(s)} \geq 0$ for $s > \tau_i$, the process $\zeta$ is increasing.
By consequence $\mathbb{E}[Q_u] = 1 - E[\zeta_u]$, $\forall u \in \mathbb{R}^+$.

□

**Comment 2.7** If the two types of construction both lead to dependence between the times of death, a difficulty concerns the choice of the best model. Indeed, it would depend on the quantities in which one is interested. For example, in the context of factor analysis of life insurance product prices, the Cox model would be better given its simplicity with closed formulas, especially in the context where the process $K$ is a subordinator However, this framework does not explain well the impact of price shocks through price jumps. This last specificity could be obtained when $K$ is a shot-noise even if the calculations would not be easy. In any case, the generalized model of Jiao and Li is a bit complicated to implement in this context because it is not easy to obtain closed forms. Moreover, it is much more interpretable in the case where the shocks do not necessarily lead to the simultaneous death of the members of the couple. A future study will focus on the latter.

# 3 Pricing of some life insurance contract under the Generalized Cox framework

Let $\tau_{(1)}$ be the moment of the first death of the couple, i.e, $\tau_{(1)} := \min(\tau^f, \tau^m)$. Setting $Z(t_1, t_2; t) := \mathbb{P}(\tau^f > t_1, \tau^m > t_2 | \mathcal{F}_t)$, for any $t_1, t_2, t \in \mathbb{R}^+$ one has $Z_{(1)}(t;t) := \mathbb{P}(\tau_{(1)} > t | \mathcal{F}_t) = Z(t,t;t)$, for any $t \in \mathbb{R}^+$ which is a supermartingale.
For simplicity, we suppose that the probability $\mathbb{P}$ is the pricing measure and the interest rate is zero.

**Definition 3.1** *A first-to-die life contract is a policy which promises a payment of an amount $R$ (that we* *suppose to be $\mathbb{F}$-predictable) at the first time of death of the spouse $\tau_{(1)}$ before a fixed time $T$. Hence, the value at time $0 \leq t \leq T$ of the death benefit is given by*

$$F_t(T) := \mathbb{E}[R_{\tau_{(1)}} \mathbb{1}_{t<\tau_{(1)}\leq T} | \mathcal{G}_t] = \mathbb{1}_{\{\tau_{(1)}>t\}} \frac{1}{Z_{(1)}(t;t)} \mathbb{E}\left[ \int_t^T R_u dA_u^{p,\tau_1} | \mathcal{F}_t \right] \tag{3.1}$$

*where $A^{p,\tau_1}$ is the predictable part of the Doob-Meyer decomposition of $Z_{(1)}$.*

The equality (3.1) is analogous to the valuation of the recovery part in a single credit default contract (see, e.g., [3, Lemma 7.4.1.2]; [2, proposition 8.2.1] for more details).

In the case where $F_t = F(t)$ and $M_t = M(t)$ and $K$ is a Lévy subordinator as given in subsection 2.1.2, one has from (2.6) that for any $t \in \mathbb{R}^+$,

$$Z_{(1)}(t;t) = e^{-F(t)} e^{-M(t)} \eta_t e^{-\phi(1)t}. \tag{3.2}$$

Hence, by using the same computations as in [11], the $\mathbb{F}$-predictable part of the Doob-Meyer decomposition of $Z_{(1)}$ is given by

$$dA_t^{p,\tau_1} = \left( \gamma^f(t) + \gamma^m(t) + \phi(1) \right) Z_{(1)}(t;t) dt, \text{ for any } t \in \mathbb{R}^+.$$

It follows from this that the value at time $0 \leq t \leq T$ of the death benefit is given by $F_t(T) = \mathbb{1}_{\{\tau_{(1)}>t\}} \widetilde{F}_t(T)$, where ( we call $\widetilde{F}$ the pre-death price)

$$\widetilde{F}_t(T) := \frac{1}{Z_{(1)}(t;t)} \int_t^T \mathbb{E}\left[ R_u \left( \gamma^f(u) + \gamma^m(u) + \phi(1) \right) Z_{(1)}(u;u) | \mathcal{F}_t \right] du. \tag{3.3}$$

By setting $R = 1$, one has

$$F_t(T) = \mathbb{1}_{\{\tau_{(1)}>t\}} \frac{1}{Z_{(1)}(t;t)} \left( \gamma^f(u) + \gamma^m(u) + \phi(1) \right) \int_t^T \mathbb{E}\left[ Z_{(1)}(u;u) | \mathcal{F}_t \right] du.$$

By consequence, simple computations lead to the so-called best estimate value of the first-to-die contract (see [14]) which is given by

$$F_0(T) = 1 - \exp\left( -F(T) - M(T) - \phi(1)T \right). \tag{3.4}$$

It is not difficult to show that $\widetilde{F}$ admits some jumps at the shock times $(\tau_i)_i$ (hence at the fatal shock time $\theta$) with non-negative jump sizes given by

$$\Delta \widetilde{F}_{\tau_i}(T) \mathbb{1}_{\{\tau_i \leq T\}} = \widetilde{F}_{\tau_i -}(T)(e^{\Delta K_{\tau_i}} - 1) \mathbb{1}_{\{\tau_i \leq T\}}. \tag{3.5}$$

**Definition 3.2** *A continuous $T$-year joint life survival benefit is a contract which promises a payment of $C$ amount at $T$ (supposed to be bounded $\mathbb{F}$-adapted) for the survival of both spouses after time $T$. Hence, its* *value at time t is then given by*

$$S_t(T) := \mathbb{E}[C_T \mathbb{1}_{\tau_{(1)}>T}|\mathcal{G}_t] = \mathbb{1}_{\{\tau_{(1)}>t\}} \frac{1}{Z_{(1)}(t;t)} \mathbb{E}\left[C_T Z_{(1)}(T;T)|\mathcal{F}_t\right] \tag{3.6}$$

**Definition 3.3** *Following [4], the value of annuity can be computed by adding life benefits. Hence, the net single premium (we denote it $\bar{A}_T$) for a T-year joint life annuity paying continuously an instantaneous benefit rate C (supposed to be $\mathcal{F}_T$-measurable) over the survival life time of both the spouses is given by*

$$\bar{A}(T) := \int_0^T S_0(u)du = \int_0^T \mathbb{E}[C_u Z_{(1)}(u;u)]du. \tag{3.7}$$

In the case where the two processes $M$ and $F$ are deterministic and $C = 1$, that value is given by

$$\bar{A}(T) = \int_0^T e^{-F(u)} e^{-M(u)} e^{-\phi(1)u} du. \tag{3.8}$$

By setting $\varphi(u) = F(u) + M(u) + \phi(1)u$, one has

$$\bar{A}(T) = \frac{1}{\varphi'(0)} - \frac{e^{-\varphi(T)}}{\varphi'(T)}.$$

# 4 Numerical application

In this section, we present some examples to illustrate how the general model built in this paper can be applied to a concrete case. We consider the generalized Cox framework in which uncertainty in the two-dimensional intensity process $(M, F)$ comes from a two-dimensional Brownian motion $W = (W_x, W_y)$, and take $\mathcal{F} = \mathcal{F}^W \vee \mathcal{F}^K$, where $\mathcal{F}^W$ is the filtration generated by $W$. Under our framework, for $t < \min(t_1, t_2)$ the joint survival function observed at time $t$ admits the following separation:

$$S_{x,y}(t_1, t_2; t) := \mathbb{P}(\tau^f > t_1, \tau^m > t_2|\mathcal{F}_t) = \tilde{S}_{x,y}(t_1, t_2; t)\mathbb{E}\left[e^{-K_{\max(t_1,t_2)}}|\mathcal{F}_t^K\right],$$

where $\tilde{S}_{x,y}(t_1, t_2; t) = \mathbb{E}\left[e^{-\left(M_{t_1}+F_{t_2}\right)}|\mathcal{F}_t^W\right]$ is the conditional joint survival function when no catastrophic events are considered. We will make use of Sibuya's function as the measure of time-dependent association of coupled lives. This function, which has been emphasized by [14], verifies the following equality

$$\mathbb{P}(\tau^f > t_1, \tau^m > t_2|\mathcal{F}_t) = \mathbb{P}(\tau^f > t_1|\mathcal{F}_t)\mathbb{P}(\tau^m > t_2|\mathcal{F}_t)\rho(t_1, t_2; t),$$

and is not restricted in $[-1,1]$ but is only strictly positive. Thus, our theoretical Sibuya's function reads:[1]

$$\rho(t_1,t_2;t) = \frac{\tilde{S}_{x,y}(t_1,t_2;t)\mathbb{E}\left[e^{-K_{\max(t_1,t_2)}}|\mathcal{F}_t^K\right]}{\mathbb{E}\left[e^{-M_{t_1}}|\mathcal{F}_t^W\right]\mathbb{E}\left[e^{-F_{t_2}}|\mathcal{F}_t^W\right]\mathbb{E}\left[e^{-\left(K_{t_1}+K_{t_2}\right)}|\mathcal{F}_t^K\right]}. \tag{4.1}$$

Moreover, under the further assumption of independence between $W_x$ and $W_y$ (and thus $\mathcal{F}^W = \mathcal{F}^{W_x} \vee \mathcal{F}^{W_y}$ with the filtrations $\mathcal{F}^{W_x}$ and $\mathcal{F}^{W_x}$ mutually independent) the Sibuya's function does not depend on the choice of the intensity process $(F,M)$, thus having

$$\rho(t_1,t_2;t) = \mathbb{E}\left[e^{K_{t_1}+K_{t_2}-K_{\max(t_1,t_2)}}|\mathcal{F}_t^K\right]. \tag{4.2}$$

## 4.1 Data and calibration procedure

We have at our disposition the well-known dataset first studied in [8].[2] The dataset consists of 14,947 joint life contracts from a Canadian company, observed during the period that goes from December 29, 1988 to December 31, 1993. We restrict our attention to the same subset of contracts analyzed in [19]. In particular we select those contracts whose males are born over the period January 1, 1907 - December 31, 1920 and females are born over the period January 1, 1910 - December 31, 1923. Since we are referring to the generation of males and females with minimum entry age of 65 and 68, respectively, from now on we indicate with $x = 65$ and $y = 68$ those generations.

Due to the censored nature of the data at hand, the classical approach described in the standard actuarial textbook to derive marginal and joint survival probabilities from mortality tables does not apply here, thus forcing us to rely on non-parametric estimators borrowed from survival analysis. In particular, we follow the recent literature ([19, 20, 29] and references therein) and use the Kaplan-Meier non-parametric estimator to reconstruct the empirical marginal survival functions $S_{65}^{Emp}(u)$, $S_{68}^{Emp}(u)$, and its two-dimensional extension called Dabrowska's ([6]) estimator for the reconstruction of the joint empirical survival function $S_{65,68}^{Emp}(s,u)$. During the last decade these estimators have become quite standard in the literature of joint mortality modelling. For this reason, we omit their constructions and refer to the relevant literature for more details. From the empirical (marginal and joint) survival functions we then construct what we call the empirical Sibuya's function, as

$$\rho_{65,68}^{Emp}(s,u) = \frac{S_{65,68}^{Emp}(s,u)}{S_{65}^{Emp}(s)S_{68}^{Emp}(u)},$$

that constitutes the starting point for the calibration procedure and is presented in Figure 1. From that figure, we note the positive quadrant dependence structure implied by the dataset under consideration.

To describe the calibration procedure used in this paper, we set $t = 0$ and define the set of parameters to be estimated as $\omega = \left(\omega^W, \omega^K\right)$, where $\omega^W$ refers to the parameters of $\tilde{S}(t,s;0)$ and $\omega^K$

[1]In our setup, we have $\mathbb{P}\left(\tau^f > s|\mathcal{F}_t\right) = \mathbb{E}\left[e^{-F_s}|\mathcal{F}_t^W\right]\mathbb{E}\left[e^{-K_s}|\mathcal{F}_t^K\right]$. Analogous calculations can be done for the marginal probability $\mathbb{P}\left(\tau^m > s|\mathcal{F}_t\right)$.

[2]The authors wish to thank the Society of Actuaries, through the courtesy of Edward (Jed) Frees and Emiliano A. Valdez, for making available to us the data in this paper.

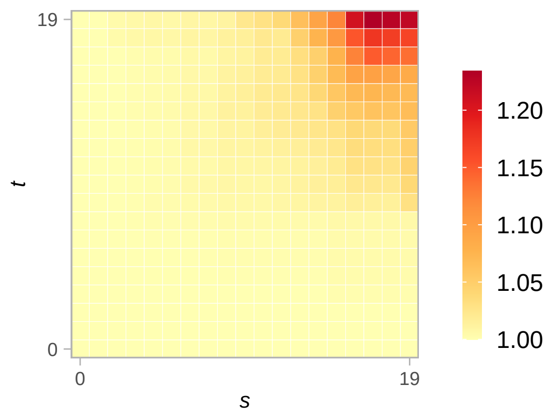


Figure 1: Heatmap of the empirical Sibuya's function $\rho^{Emp}_{65,68}(s,t)$ .

refers to the parameters of the catastrophic process $K$; As a matter of notation, we use $\rho(\omega^W, \omega^K; t, s)$ to denote the theoretical Sibuya's function evaluated at $t = 0$ when the values of the parameters of the model are $(\omega^W, \omega^K)$. We use the following two-step procedure, where the first step is needed when additional dependence other than joint death due to catastrophic events is assumed:

1. Estimate the parameters of theoretical joint survival function $\tilde{S}(t_1, t_2; 0)$; call the estimated parameters $\hat{\omega}^W$.

2. Define the objective function as the sum of squared differences between the theoretical and empirical Sibuya's function, that is

$$l(\omega^K) = \sum_{t=0}^{N} \sum_{s=0}^{M} \left(\rho(\hat{\omega}^W, \omega^K; t, s) - \rho^{Emp}_{65,68}(t,s)\right)^2 .$$

Then, the calibrated values of the disaster process are defined to be $\hat{\omega}^K := \arg\min_{\omega^K} l((\omega^K))$.

We make two remarks on the first step of the calibration procedure. First, it is only needed when additional dependence structure other than the Marshall-Olkin type is assumed in the model. Second, it corresponds to a standard estimation procedure of the joint survival function. The models based on our approach can be built on top of existing model, thus exploiting the properties and the existing estimation procedures. We view this feature of our modelling approach as an additional advantage.

## 4.2 Models specifications

We make use of the setup above to calibrate the Sibuya's function that captures the dependence structure of the joint mortality phenomenon. We consider two possible alternatives, both subordinators, for the disaster process $K$:

- In the first alternative, $K$ is Compound Poisson Process with non-negative jumps, i.e., $K_t = \sum_{i=1}^{N_t} D_i, \forall t \geq 0$ where $N$ is a Poisson process with intensity $\lambda$ where $(D_i, i \geq 1)$ are i.i.d. random variables, all exponentially distributed with parameter $\gamma$, and independent from $N$. In this case, we have

$$\mathbb{E}\left[e^{-K_u}|\mathcal{F}_t^K\right] = \eta_t^P e^{-\frac{\lambda u}{\gamma+1}}, \text{ for } u \geq t.$$

  For practical purposes, when calibrating the compound Poisson process for $K$, we use the following function:

$$g_{CP}(u;\eta,\delta) = \eta e^{-\delta u},$$

  $\eta, \gamma > 0$, so that $\omega^K = (\eta, \delta)$, with $\delta = \frac{\lambda}{\gamma+1}$ to avoid identification issues between $\lambda$ and $\gamma$.

- The second alternative is a Gamma process, with Lévy measure $\nu(x) = \frac{\delta e^{-\lambda x}}{x}$, where $\delta > 0$ governs the arrivals of jumps and reciprocal of $\lambda > 0$ models the size of jumps. In this case, we have

$$\mathbb{E}\left[e^{-K_u}|\mathcal{F}_t^K\right] = \eta_t^G \left(1 + \frac{1}{\lambda}\right)^{-\delta u}, \text{ for } u \geq t.$$

  For the actual calibration, we use the following function:

$$g_{Gamma}(u;\eta,\delta) = \eta^{-\delta u},$$

  with $\eta = 1 + \frac{1}{1+\lambda} > 1$, $\delta > 0$, and $\omega^K = (\eta, \delta)$. We have set the initial level $\eta_t^G = 1$ to avoid identification issues with the parameter $\eta$.

To complete the specification of the models to be calibrated, we need to specify the functional form of $\tilde{S}_{xy}(u, s, ; t)$. Again, we consider two cases. In the first case, we assume independence between $M$ and $F$, so that the theoretical Sibuya's function does not depend on any assumption about marginal mortality, as equation (4.2) shows. Although this assumption seems to be too restrictive, it constitutes a useful benchmark for evaluating the appropriateness of more general models. In the second case, we follow [19], so that

$$\tilde{S}_{xy}(u, s; t) = C\left(S_x(u;t), S_y(s;t)\right), \tag{4.3}$$

being $C(\cdot, \cdot)$ an Archimedean copula function [24]. The marginal mortality models for males and

| $\mu_F(0)$ | $\mu_M(0)$ | $a_F$ | $a_M$ | $\sigma_F$ | $\sigma_M$ | $\theta$ |
|---|---|---|---|---|---|---|
| 0.00469 | 0.0204 | 0.1249 | 0.081 | 0.00024 | 0.0034 | 0.72003 |

Table 1: Estimated values of $\omega^W$ as reported in [19].

females are instead defined, for $h = x, y$ and $u \geq t$, as:[3]

$$S_h(u;t) = e^{\mu_h(t)\beta_h(u-t)}, \tag{4.4}$$

where $\beta_h(u) = \frac{1-e^{-b_h t}}{c_h + d_h e^{b_h t}}$ and $b = -\sqrt{a_h^2 + 2\sigma_h^2}$, $c_h = \frac{b_h + a_h}{2}$ and $d_h = \frac{b_h - a_h}{2}$.

The specifications above for the disaster process $K$ and the joint survival function $\tilde{S}(u, s; t)$ lead to four different models

- In model $M_1$, we assume independence between $F$ and $M$, so that the theoretical Sibuya's function does not depend on the choice of the marginal survival functions. In addition, we assign the compound Poisson process to $K$;
- In model $M_2$ we keep the assumption of independence between $F$ and $M$, but use the Gamma process to model catastrophic events;
- In model $M_3$, we introduce a more complex form of dependence for the joint survival function. In particular, we use (4.3) for the specification of the joint survival probability not considering catastrophic event, with associated copula function[4]

  $$C(r, v) = \left(\ln\left(e^{r^{-\theta}} + e^{v^{-\theta}} - e\right)\right)^{-\frac{1}{\theta}}.$$

  The marginal survival probabilities are modeled in line with (4.4). We use the the compound Poisson process to model $K$;
- Finally, model $M_4$ follows $M_3$ to model $\tilde{S}_{x,y}(\cdot, \cdot)$, but uses the Gamma process to represent the dynamics of $K$.

For the calibration of models $M_3$ and $M_4$, the first step of the calibration procedure described in subsection 4.1 is required. This involves the estimation of all the parameters used in the modelling framework of $\tilde{S}(\cdot, \cdot, 0)$, that is $\omega^W = (\mu_F(0), \mu_M(0), a_F, a_M, \sigma_F, \sigma_M, \theta)$. The $\omega^W$ can be estimated through the standard procedure described in [19], which in turn uses the Wang and Well ([28]) procedure for the estimation of the copula parameter $\theta$. In the context of this paper, as we use the same sample used in [19] we skip the estimation $\omega^W$ and uses the estimated parameters that can be found there. For the reader's convenience, we report in Table 1 the estimated values.

[3]This functional form of the marginal survival probability comes form assuming a stochastic intensity of the form $d\mu_h(u) = a_h\mu_h(u) + \sigma_h\sqrt{\mu_h(h)}dW_h(u)$, with $a, \sigma > 0$. A sufficient condition for $S_h(u;t)$ to be a valid survival function is $\sigma^2 < 2dc$. Additional details about this model can be found in [19, 21].

[4][19] refers to this model of association as the *4.2.20 Nelsen* copula function. Originally proposed in [24], a detailed study can be found in [27]. The choice of this particular model of association is due to the fact that it produces the best fit in a range of several Archimedean copulas for the data used in this paper ([19]).

### 4.3 Calibration results

We report the calibrated parameters together with the sum of squared error for models $M_1$-$M_4$ in Table 2. Figure 2 reports the heatmaps of the theoretical survival functions, while Figure 3 shows the heatmaps of the relative errors for each model. The relative error for model $m$ is defined as $e(t,s) = \frac{\rho^{Emp}_{65,68}(t,s) - \rho^m(t,s)}{\rho^{Emp}_{65,68}(t,s)}$, where $\rho^m(t,s)$ is the theoretical Sibuya's function of model $m$ computed at the calibrated parameters.

| Model | Parameters | | |
|---|---|---|---|
| | $\eta$ | $\delta$ | Objective value |
| $M_1$ | 0.981414 | 0.005973 | 0.205518 |
| $M_2$ | 1.003593 | 1.106367 | 0.25250423 |
| $M_3$ | 0.978167 | 0.000454 | 0.15182141 |
| $M_4$ | 1.008138 | 1.601355 | 0.07105885 |

Table 2: Calibrated parameters and objective values for models $M_1$-$M_4$.

Our numerical results show the superior performance of models $M_3$ and $M_4$ incorporating dependence due to both the common style of life of spouses through copulas and to some catastrophic event that causes death of both annuitants with respect to models $M_1$ and $M_2$ that consider only the second type of dependence. This is in accordance with recent empirical evidence on the same dataset ([9, 10]). Our results also suggest that, for the data at hand, the Gamma process produces a more accurate approximation of empirical Sibyua's function than the Compound Poisson process.

## 5 Conclusions

In this paper, we build on the recent literature on credit risk to develop two joint mortality modelling frameworks that incorporate dependence due to a common fatal event that causes death of both annuitants. Our approach has an important advantage. Differently to the vast majority of stochastic mortality models, in our setup the times of death are covered by a sequence of fatal shock times. The pricing of life-insurance product can be conducted with relative ease. Another advantage over the recent literature is that our framework accommodates stochastic mortality quite naturally.

In the numerical application we show how to construct models belonging to our classes and how to calibrate them. This application shows how existing models can be extended to incorporate dependence due to a common catastrophic events. Calibration of the process that represents the fatal event can be performed straightforwardly. In terms of calibration performance, our approach turns out to possess a more than satisfactory goodness of fit. For the data at hand, models that includes dependence with copula and Marshall-Olkin-type dependence perform best and the Gamma process produces more accurate results than the Compound Poisson process.

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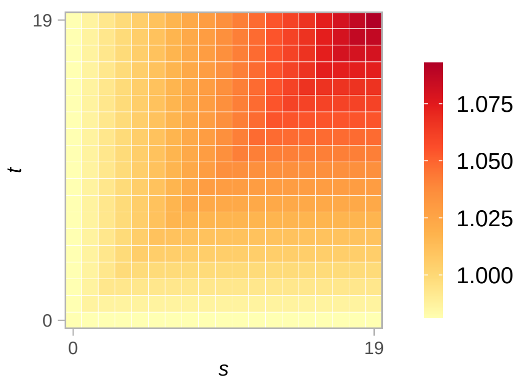
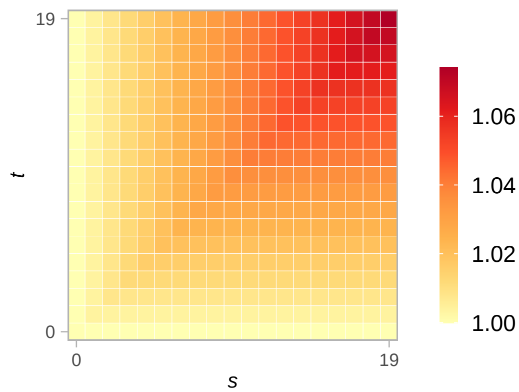
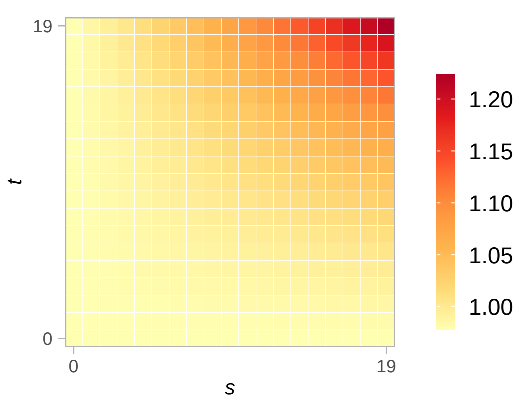


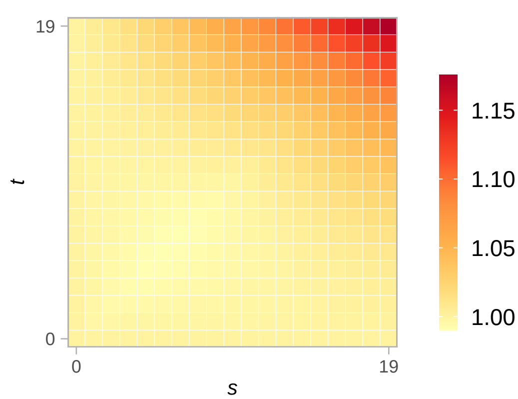

(a) $M_1$




(b) $M_2$

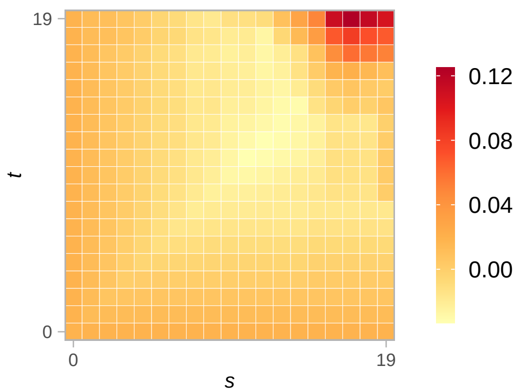



(c) $M_3$

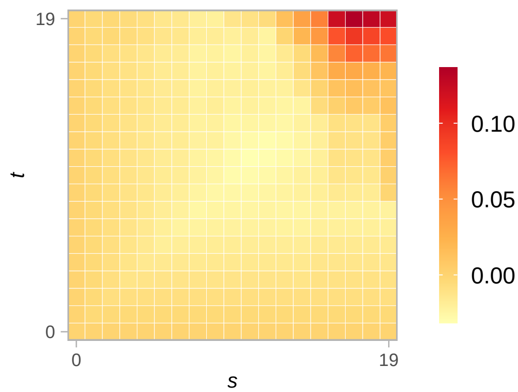



(d) $M_4$

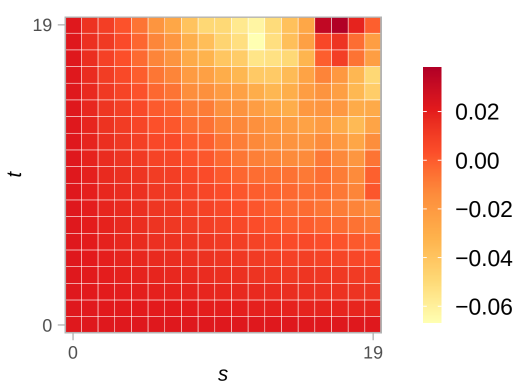
Figure 2: Theoretical joint survival functions for models $M_1$-$M_4$.



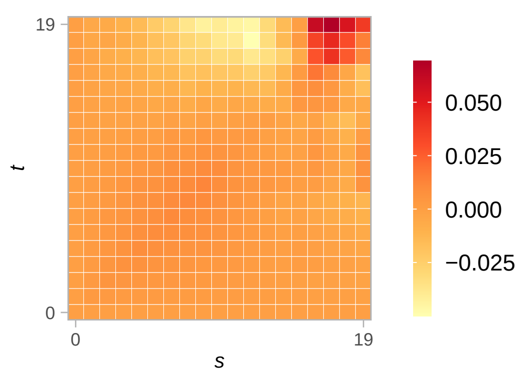

(a) $M_1$




(b) $M_2$




(c) $M_3$




(d) $M_4$

Figure 3: Heatmap of relative errors for models $M_1$-$M_4$.